

CONGRUENCE RELATIONS FOR SHIMURA VARIETIES ASSOCIATED TO $GU(n-1, 1)$

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ABSTRACT. We prove the congruence relation for the mod- p reduction of Shimura varieties associated to a unitary similitude group $GU(n-1, 1)$ over \mathbb{Q} , when p is inert and n odd. The case when n is even was obtained by T. Wedhorn and O. Bültel in [Bültel and Wedhorn(2006)], as a special case of a result of B. Moonen in [Moonen(2004)], when the ordinary locus of the p -isogeny space is dense. This condition fails in our case. A key element is the understanding of the supersingular locus, for which we refer to the two articles [Volleard(2010), Volleard and Wedhorn(2011)]. The proof makes extensive use of elementary algebraic geometry, but also some deeper results.

INTRODUCTION

Let (G, X) be a Shimura datum where G is a reductive group over \mathbb{Q} . We fix a prime p . For every compact open subgroup $K \subset G(\mathbb{A}_f)$, let Sh_K the associated Shimura variety with reflex field E . The complex points of Sh_K are:

$$Sh_K(\mathbb{C}) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f) / K)$$

When K is sufficiently small, Sh_K is smooth. Assume $G_{\mathbb{Q}_p}$ is unramified and $K = K_p K^p$ with $K_p \subset G(\mathbb{Q}_p)$ hyperspecial and $K^p \subset G(\mathbb{A}_f^p)$, then Sh_K is said to have good reduction at p . Let \mathfrak{p} be a prime in E lying over p . In [Blasius and Rogawski(1994)], the authors define a polynomial $H_{\mathfrak{p}}$ with coefficients in the Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p)/K_p)$, the set of \mathbb{Q} -linear combinations of K_p -double cosets of $G(\mathbb{Q}_p)$. It is made into a ring by convolution. This ring acts on the cohomology of the Shimura variety. Denote by $Fr_{\mathfrak{p}}$ the conjugacy class of geometric Frobenius in $\text{Gal}(\overline{\mathbb{Q}}/E)$. Blasius and Rogawski conjectured the following:

Conjecture 1. *Let ℓ be a prime $\neq p$. Inside the ring $\text{End}_{\mathbb{Q}_\ell}(H_{et}^\bullet(Sh_K \times_E \overline{\mathbb{Q}}, \mathbb{Q}_\ell))$, the following relation holds*

$$H_{\mathfrak{p}}(Fr_{\mathfrak{p}}) = 0$$

This equation makes sense since the action of Galois commutes with that of $\mathcal{H}(G(\mathbb{Q}_p)/K_p)$. In the PEL case, an integral model over $\mathcal{O}_{E_{\mathfrak{p}}}$ can be defined explicitly, and the cohomology of $\overline{Sh}_K = Sh_K \times \kappa(\mathcal{O}_{E_{\mathfrak{p}}})$ coincides in many cases with that of $Sh_K \times E$.

In the case of Shimura curves, this conjecture was proved by Eichler, Shimura and Ihara, and was used to determine completely the eigenvalues of $Fr_{\mathfrak{p}}$ acting on $H_{et}^i(Sh_K \times_E \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$. More general situations have been dealt with: T. Wedhorn proved conjecture 1 in the PEL case for groups that are split over \mathbb{Q}_p in [Wedhorn(2000)], O. Bültel for certain orthogonal groups in [Bültel(2002)], and they worked out together the unitary case of signature $(n-1, 1)$ with n even in [Bültel and Wedhorn(2006)].

In the latter article, the authors use a moduli space $p\text{-}\mathcal{I} \text{ sog}$ which parametrizes p -isogenies between points of Sh_K . It was first introduced in [Faltings and Chai(1990)]. It comes with two maps s, t to Sh_K , associating an isogeny to its source and target respectively. For any field L with a map $\mathcal{O}_{E_{\mathfrak{p}}} \rightarrow L$, we consider the \mathbb{Q} -algebra of cycles in $p\text{-}\mathcal{I} \text{ sog} \times L$, where multiplication is defined by composition of isogenies, and we denote by $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog}^{ord} \times L]$ the subalgebra generated by the irreducible components.

In [?] and [Moonen(2004)], the authors define the μ -ordinary locus in the good reduction of a PEL Shimura variety. It is at the same time a Newton polygon stratum and an Ekedahl-Oort stratum. Furthermore, it possesses a unique isomorphism class of p -divisible groups. It can also be defined as the unique open stratum in each of these stratifications. We will denote by \overline{Sh}_K^{ord} the ordinary locus. Define the

ordinary locus $p\text{-}\mathcal{I} \text{ sog}^{ord} \times \kappa(\mathcal{O}_{E_p})$ of $p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})$ by taking inverse image by s (or t). Finally define $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog}^{ord} \times \kappa(\mathcal{O}_{E_p})]$ in the same fashion as hereabove. We have a diagram of \mathbb{Q} -algebra homomorphisms:

$$\begin{array}{ccc}
\mathcal{H}_0(G(\mathbb{Q}_p)//K_p) & \xrightarrow{h^0} & \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times E] \\
\downarrow \dot{S} & & \downarrow \sigma \\
& & \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})] \\
& & \uparrow \text{cl} \quad \downarrow \text{ord} \\
\mathcal{H}_0(M(\mathbb{Q}_p)//(K_p \cap M(\mathbb{Q}_p))) & \xrightarrow{\bar{h}} & \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog}^{ord} \times \kappa(\mathcal{O}_{E_p})]
\end{array}$$

The big square is commutative. Here $M \subset G_{\mathbb{Q}_p}$ is the centralizer of the norm of the minuscule coweight μ of G associated to (G, X) . The algebras on the left hand side of the diagram are subalgebras of the Hecke algebras containing functions with integral support. The morphism \dot{S} is a twisted version of the Satake homomorphism. The map σ is specialization of cycles, the map ord intersects a cycle in $p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})$ with the ordinary locus, and the map cl is simply defined by taking the closure of a cycle of $p\text{-}\mathcal{I} \text{ sog}^{ord} \times \kappa(\mathcal{O}_{E_p})$. There is a natural Frobenius section of s defined on $\overline{Sh_K}$, defined by mapping an abelian variety to its Frobenius isogeny, which produces a closed subscheme F of $p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})$. This subscheme is ordinary, in the sense that $\text{cl} \circ \text{ord}(F) = F$. In this context, by “congruence relation” we mean the conjecture:

Conjecture 2. *Consider the polynomial H_p inside $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})]$ via the morphism $\sigma \circ h^0$. The element F lies in the center of this ring and the following relation holds:*

$$H_p(F) = 0$$

It is related to conjecture 1 by using functorial properties of cohomology. The geometric relation $H_p(F) = 0$ implies the same equality on the cohomology. B. Moonen has shown the “ordinary” congruence relation:

Theorem 3. *Consider the polynomial H_p inside $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog}^{ord} \times \kappa(\mathcal{O}_{E_p})]$ via the morphism $\text{ord} \circ \sigma \circ h^0$. In this ring, the following relation holds:*

$$H_p(F) = 0$$

When the ordinary locus of $p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})$ is dense, this theorem is equivalent to the congruence relation. This condition is satisfied in almost all the examples where conjecture 2 is known. In the unitary similitude case, this density condition is satisfied if and only if n is even. As Moonen points out, the condition fails for Hilbert-Blumenthal varieties. However, conjecture 2 can be proved in this case by the following argument. The Hecke polynomial factors into a product $H_p = PQ$ such that P annihilates F on the ordinary locus, and P is ordinary, that is $\text{cl} \circ \text{ord}(P) = P$. The result follows immediately.

In this article, we prove conjecture 2 in the unitary similitude case $GU(n-1, 1)$ when n is odd. We first show that the Hecke polynomial factors into a product $H_p(t) = R(t) \cdot (t - p^{n-1}1_{pK_p})$. Let H'_p and R' be the polynomials obtained by applying the map $\text{cl} \circ \text{ord}$ to H_p and R . We get: $(H_p - H'_p)(t) = (R - R')(t) \cdot (t - p^{n-1}\langle p \rangle)$, where $\langle p \rangle$ is the multiplication-by- p cycle. It is the image of 1_{pK_p} by $\sigma \circ h^0$. Because of theorem 3, we have $H'_p(F) = 0$ inside $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})]$. Therefore, conjecture 2 boils down to $(H_p - H'_p)(F) = 0$. The factor $(R - R')(F)$ is a cycle formed only of supersingular components of $p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})$. The final argument is the following result:

Theorem. *Let $C \subset p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})$ be a supersingular irreducible component. Then:*

$$C \cdot (F - p^{n-1}\langle p \rangle) = 0$$

inside $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})]$.

We will now give an overview on how this article is organized. In the first section, we establish the factorization of the Hecke polynomial. In the second one, we give the main facts concerning the Shimura variety. All of them can be found in [Bültel and Wedhorn(2006)], we choose to repeat them here for the sake of clarity. Section 3 is dedicated to the moduli space of p -isogenies. Section 4 studies the supersingular

locus of $p\text{-}\mathcal{I}so \times \kappa(\mathcal{O}_{E_p})$. Here we use mainly [Vollaard(2010), Vollaard and Wedhorn(2011)] and also [Bültel and Wedhorn(2006)] for some key results. Finally, in section 5, we prove conjecture 2.

NOTATIONS

- (1) We fix $n \geq 3$ an **odd integer**, $k = \frac{n+1}{2}$ and $p > 2$ a prime. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of \mathbb{Q}_p . We denote by $\overline{\mathbb{Q}}$ the algebraic closure of \mathbb{Q} inside \mathbb{C} . We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.
- (2) Let E be an imaginary quadratic extension of \mathbb{Q} , such that p is **inert** in E . We write $\sigma : x \mapsto \overline{x}$ for the non trivial automorphism of E and E_p for the completion of E at p . Let \mathcal{O}_{E_p} be the ring of integers of E_p and $\kappa(\mathcal{O}_{E_p}) = \frac{\mathcal{O}_{E_p}}{p\mathcal{O}_{E_p}}$ the residual field.
- (3) We fix an embedding $\vartheta : E \hookrightarrow \overline{\mathbb{Q}}$. We denote by $\overline{\mathbb{F}}$ the algebraic closure of $\kappa(\mathcal{O}_{E_p})$ provided by the embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$. We choose an element $\alpha \in E^\times \cap \mathcal{O}_{E_p}^\times$ such that the imaginary part of α is > 0 and $\alpha + \overline{\alpha} = 0$. If $z \in \mathbb{C}$ and $z = a + \alpha b$, $a, b \in \mathbb{R}$, we call b the α -imaginary part of z .
- (4) (V, ψ) is a hermitian space of dimension n , i.e V is an n -dimensional E -vector space, and $\psi : V \times V \rightarrow E$ a non degenerate hermitian pairing. **We assume the signature of (V, ψ) to be $(n-1, 1)$.**
- (5) G is the (connected, reductive) algebraic \mathbb{Q} -group of unitary similitudes of (V, ψ) .
- (6) Let $\mathcal{B}_W = (e_1, \dots, e_n)$ be a Witt basis of $V \otimes \mathbb{Q}_p$. This means $V \otimes \mathbb{Q}_p = V_0 \oplus \bigoplus_{1 \leq i \leq k} H_i$ is an orthogonal Witt decomposition, where $V_0 = \text{Vect}_{E_p}(e_k)$ is anisotropic, $H_i = \text{Vect}_{E_p}(e_i, e_{n+1-i})$ is a hyperbolic plane with $\psi(e_i, e_{n+1-i}) = 1$.
- (7) In the basis \mathcal{B}_W , the diagonal matrices of $G_{\mathbb{Q}_p}$ form a torus T and the upper-triangular matrices of G a Borel subgroup B containing T . In the basis \mathcal{B} , an element of $T(\mathbb{Q}_p)$ has matrix $\text{diag}(x_1, \dots, x_n) \in GL_n(E_p)$ with

$$\overline{x_1}x_n = \overline{x_2}x_{n-1} \dots = \overline{x_k}x_k.$$

- (8) Let $\Omega(T)$ be the Weyl group of T over \mathbb{Q}_p . It is the group of permutations of $\{1, \dots, n\}$ fixing the equations above. Thus,

$$\Omega(T) = \{\sigma \in \mathfrak{S}_n, \forall i \in \{1, \dots, n\}; \sigma(i) + \sigma(n+1-i) = n\}.$$

- (9) Let ρ be the half-sum of positive roots with respect to (B, T) .
- (10) Let Λ be the \mathcal{O}_{E_p} -lattice generated by the e_i . **We assume that ψ defines a perfect pairing**
 $\Lambda \times \Lambda \rightarrow \mathcal{O}_{E_p}$. This amounts to $\psi(e_k, e_k) \in \mathbb{Z}_p^\times$ and implies that $\det(\psi) = 1 \in \frac{\mathbb{Q}_p^\times}{N(E_p^\times)}$.
- (11) Let $K_p = \text{Stab}_{G(\mathbb{Q}_p)}(\Lambda)$, it is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. We write $L = K_p \cap B(\mathbb{Q}_p)$ and $T_c = K_p \cap T(\mathbb{Q}_p)$.
- (12) Let $\varphi : V \times V \rightarrow \mathbb{Q}$ be the α -imaginary part of ψ . Then φ is a skew-symmetric form such that $\forall e \in E, \forall x, y \in V, \varphi(ex, y) = \varphi(x, \overline{e}y)$.

1. HECKE POLYNOMIAL

1.1. Unitary similitude group. There is an isomorphism $V \otimes_{\mathbb{Q}} E \simeq \bigoplus_{\tau \in \text{Gal}(E/\mathbb{Q})} V$. The choice of $\text{id} \in \text{Gal}(E/\mathbb{Q})$ gives an isomorphism:

$$(1.1) \quad G_E \simeq GL_E(V) \times \mathbb{G}_m.$$

Let \mathcal{B} be an E -basis of V and let J be the matrix of ψ in \mathcal{B} . The group $\text{Gal}(E/\mathbb{Q}) = \{1, \sigma\}$ acts on $G(E) \simeq GL_n(E) \times E^\times$ by:

$$(1.2) \quad \sigma.(A, \lambda) = (\overline{\lambda}J({}^t\overline{A}^{-1})J, \overline{\lambda}), \quad \forall (A, \lambda) \in GL_n(E) \times E^\times.$$

1.2. Dual group. For the diagonal torus $T_{n, \mathbb{Q}} \subset GL_{n, \mathbb{Q}}$, we denote by χ_1, \dots, χ_n (resp. μ_1, \dots, μ_n) the usual characters (resp. cocharacters) of $T_{n, \mathbb{Q}}$. Let χ_0 (resp. μ_0) the character (resp. cocharacter) of $T_{n, \mathbb{Q}} \times \mathbb{G}_{m, \mathbb{Q}}$ defined by $(A, x_0) \mapsto x_0$ (resp. $x \mapsto (I_n, x)$).

The dual group of G is $\widehat{G} = GL_{n, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}$. A splitting is a triplet $\Sigma = (\widehat{T}, \widehat{B}, \{X_\alpha\})$ where $(\widehat{B}, \widehat{T})$ is a Borel pair of \widehat{G} and $X_\alpha \in \text{Lie}(\widehat{G})$ an eigenvector for every simple root α of \widehat{G} . The $\text{Gal}(E/\mathbb{Q})$ -action on

$\Psi(\widehat{G}) = \Psi(G)^\vee$ lifts uniquely to an automorphism of \widehat{G} fixing Σ (cf. [Blasius and Rogawski(1994)], section 1.6). We make the following standard choices:

$$\begin{aligned}\widehat{T} &= \{\text{diagonal matrices}\} \times \mathbb{G}_{m,\mathbb{C}} \\ \widehat{B} &= \{\text{upper-triangular matrices}\} \times \mathbb{G}_{m,\mathbb{C}} \\ \{X_k\} &= (\delta_{i,k}\delta_{j,k+1}) \text{ for } k = 1, 2, \dots, n-1.\end{aligned}$$

The vector X_k lies in $\text{Lie}(\widehat{G}) = M_n(\mathbb{C}) \oplus \mathbb{C}$ and is an eigenvector for the simple root $\chi_k - \chi_{k+1}$ of \widehat{T} . There is a unique non trivial automorphism of \widehat{G} fixing Σ , giving the action of σ on \widehat{G} :

$$(1.3) \quad \begin{array}{ccc} \widehat{G} & \longrightarrow & \widehat{G} \\ (A, \lambda) & \longmapsto & (J'({}^t A^{-1})J', \det(A)\lambda) \end{array}$$

where $J' = ((-1)^{i-1}\delta_{i,n+1-j})_{i,j}$ (cf. *ibid*, 1.8(c)).

The choice of the basis \mathcal{B}_W gives an identification between $(G_{\overline{\mathbb{Q}_p}}, T_{\overline{\mathbb{Q}_p}})$ and $(GL_{n,\overline{\mathbb{Q}_p}} \times \mathbb{G}_{m,\overline{\mathbb{Q}_p}}, T_{n,\overline{\mathbb{Q}_p}} \times \mathbb{G}_{m,\overline{\mathbb{Q}_p}})$ through (1.1). We fix the identification $\Psi(\widehat{G}, \widehat{T}) \simeq \Psi(G, T)^\vee$ given by $\chi_i \leftrightarrow \mu_i$ for $i = 0, \dots, n$. We also identify \widehat{T} and $\text{Hom}(X_*(T), \mathbb{C}^\times)$ such that $(\text{diag}(x_1, \dots, x_n), x_0) \in \widehat{T}$ corresponds to the map $\mu_i \mapsto x_i$.

1.3. Shimura datum. Choose an isomorphism $G_{\mathbb{R}} \simeq G_{J_0, \mathbb{R}}$. Consider the morphism $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ of algebraic groups over \mathbb{R} defined on \mathbb{R} -points by:

$$(1.4) \quad \begin{array}{ccc} \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times & \longrightarrow & G(\mathbb{R}) \simeq G_{J_0}(\mathbb{R}) \\ z & \longmapsto & \text{diag}(z, \dots, z, \bar{z}) \end{array}.$$

Let X be the $G(\mathbb{R})$ -conjugacy class of h . Then (G, X) is a Shimura datum ([Milne(2005)] definition 5.5). Its reflex field is E . Composing $h_{\mathbb{C}}$ on the right-hand side by $\mathbb{G}_{m,\mathbb{C}} \hookrightarrow \prod_{\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_{m,\mathbb{C}} \simeq \mathbb{S}_{\mathbb{C}}$ (given by $\sigma = \text{Id}$) gives a cocharacter $\mu : \mathbb{G}_{m,\mathbb{C}} \rightarrow G_{\mathbb{C}}$. Finally, write $\widehat{\mu}$ for the associated character of \widehat{T} which is dominant relative to \widehat{B} . We have $\widehat{\mu} = \chi_1 + \dots + \chi_{n-1} + \chi_0$.

1.4. The representation r . Let r be the irreducible representation of $GL_{n,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ of highest weight $\widehat{\mu}$ relatively to $(\widehat{B}, \widehat{T})$. Let ρ denote the identity representation $GL_{n,\mathbb{C}} \rightarrow GL_{n,\mathbb{C}}$, its weights are $(\chi_i)_{1 \leq i \leq n}$. The representation $\det \otimes \rho^\vee$ of $GL_{n,\mathbb{C}}$ is irreducible and its highest weight is $\chi_1 + \dots + \chi_{n-1} = \det - \chi_n$. Thus, we can define r as follows:

$$r : \begin{array}{ccc} GL_{n,\mathbb{C}} \times \mathbb{G}_m & \longrightarrow & GL_{n,\mathbb{C}} \\ (A, \lambda) & \longmapsto & \lambda \det(A) {}^t A^{-1} \end{array}.$$

Definition 4. The Hecke polynomial associated to (G, X) is:

$$(1.5) \quad H_p(t) = \det(t - p^{n-1}r(g(\sigma \cdot g))).$$

The coefficients of H_p are functions on \widehat{G} invariant under twisted conjugation $c_x : g \mapsto xg(\sigma \cdot x^{-1})$, for $x \in \widehat{G}$.

1.5. Hecke algebra.

Definition 5. For any \mathbb{Q} -algebra R , the Hecke algebra $\mathcal{H}_R(G(\mathbb{Q}_p)//K_p)$ is the set of K_p -biinvariant, compactly supported functions $G(\mathbb{Q}_p) \rightarrow R$. Multiplication is defined by convolution:

$$(f \star g)(y) = \int_{G(\mathbb{Q}_p)} f(x)g(x^{-1}y)dx$$

where the Haar measure on $G(\mathbb{Q}_p)$ is normalized by $|K_p| = 1$.

We recall some facts about the Satake isomorphism. We identify $\mathbb{Q}[X_*(A)]$ and $\mathcal{H}_{\mathbb{Q}}(T(\mathbb{Q}_p)//T_c)$ by $\lambda \mapsto 1_{\lambda(p)T_c}$ for $\lambda \in X_*(A)$. In [Wedhorn(2000)] (1.7,1.8), the twisted Satake homomorphism \dot{S}_T^G is defined by the composition

$$\mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p)//K_p) \rightarrow \mathcal{H}_{\mathbb{Q}}(B(\mathbb{Q}_p)//L) \rightarrow \mathcal{H}_{\mathbb{Q}}(T(\mathbb{Q}_p)//T_c)$$

where the first arrow is restriction of functions and the second is the quotient by the unipotent radical of B . It induces an isomorphism between $\mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p)//K_p)$ and the subalgebra of $\mathcal{H}_{\mathbb{Q}}(T(\mathbb{Q}_p)//T_c)^{\Omega(T), \bullet}$ (the Weyl group acts by the “dot action”, see *ibid*, 1.8). Denote by $S_T^G : \mathcal{H}_{\mathbb{C}}(G(\mathbb{Q}_p)//K_p) \rightarrow \mathcal{H}_{\mathbb{C}}(T(\mathbb{Q}_p)//T_c)$ the usual Satake isomorphism. Then $S_T^G = \alpha \circ \dot{S}_T^G$ where $\alpha : \mathbb{C}[X_*(A)] \rightarrow \mathbb{C}[X_*(A)]$ is defined by $\nu \mapsto p^{-2\langle \rho, \nu \rangle} \nu$.

1.6. Hecke polynomial. The coefficients of H_p are polynomial functions on \widehat{G} invariant under twisted conjugation. This is the same as polynomial functions on \widehat{T} invariant under $\Omega(T)$ and twisted conjugation, or polynomial functions on \widehat{A} invariant under $\Omega(T)$. By the untwisted Satake isomorphism, they correspond to elements in $\mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p)//K_p)$.

Lemma 6. *The function $\widehat{G} \rightarrow \mathbb{C}$ given by $(A, x) \mapsto \det(A)x^2$ is invariant under twisted conjugation. It corresponds to the element 1_{pK_p} in $\mathcal{H}_{\mathbb{C}}(G(\mathbb{Q}_p)//K_p)$.*

Proof. The element 1_{pK_p} maps to 1_{pT_c} by the Satake isomorphism. This element corresponds to $\lambda \in \mathbb{Q}[X_*(A)]$ where λ is the cocharacter $u \mapsto u \cdot \text{Id}$. Using the identification (1.1), we have $\lambda = \sum_{i>0} \mu_i + 2\mu_0$. The associated character of \widehat{T} is $\sum_{i>0} \chi_i + 2\chi_0$, which is the function $(A, x_0) \mapsto \det(A)x_0^2$. \square

Let $g = (A, x_0) \in \widehat{T}$, with $A = \text{diag}(x_1, \dots, x_n)$. Then $r(g(\sigma \cdot g)) = \det(A)x_0^2 \text{diag}(\frac{x_n}{x_1}, \dots, \frac{x_1}{x_n})$, and:

$$\begin{aligned} H_p(t) &= \det(t - p^{n-1}r(g(\sigma \cdot g))) \\ &= \prod_{i=1}^n \left(t - p^{n-1} \det(A)x_0^2 \frac{x_{n+1-i}}{x_i} \right) \\ &= R(t) \times (t - p^{n-1} \det(A)x_0^2) \end{aligned}$$

where $R(t) = \prod_{i \neq k} \left(t - p^{n-1} \det(A)x_0^2 \frac{x_{n+1-i}}{x_i} \right)$. The polynomial R is invariant under $\Omega(T)$ and twisted conjugation. We deduce the following result.

Theorem 7. *The Hecke polynomial H_p in $\mathcal{H}(G(\mathbb{Q}_p)//K_p)$ factors into a product*

$$H_p(t) = R(t) \cdot (t - p^{n-1} 1_{pK_p})$$

where $R(t) \in \mathcal{H}(G(\mathbb{Q}_p)//K_p)[t]$.

2. THE SHIMURA VARIETY

2.1. The moduli problem. Let $K^p \subset G(\mathbb{A}_f^p)$ be a compact open subgroup and denote by Sh_K the moduli space associated to the data $(E, \sigma, V, \psi, \mathcal{O}_{E, (p)}, \Lambda, h, \mu)$ according to Kottwitz [Kottwitz(1992)]. We assume K^p to be sufficiently small such that this moduli problem is representable by a smooth, quasi-projective scheme over \mathcal{O}_{E_p} . For any noetherian \mathcal{O}_{E_p} -scheme S , it classifies the following data, up to prime-to- p -isogeny:

- (1) An abelian scheme A of dimension n over S .
- (2) A \mathbb{Q} -homogeneous polarization $\bar{\lambda} = \mathbb{Q}\lambda$ for some prime-to- p polarization λ .
- (3) An action $\iota : \mathcal{O}_E \otimes \mathbb{Z}_{(p)} \hookrightarrow \text{End}(A) \otimes \mathbb{Z}_{(p)}$ compatible with $\bar{\lambda}$.
- (4) A $\pi_1(S, s)$ -stable K^p -orbit of compatible isomorphisms $\bar{\eta} : V(\mathbb{A}_f^p) \xrightarrow{\sim} H_1(A_s, \mathbb{A}_f^p)$ for one geometric point s in each connected component of S .

Further, $(A, \iota, \bar{\lambda}, \bar{\eta})$ satisfies the determinant condition: the characteristic polynomial of $e \in \mathcal{O}_E \otimes \mathbb{Z}_{(p)}$ acting on $\text{Lie}(A)$ is $(T - e)^{n-1}(T - \bar{e}) \in \mathcal{O}_S[T]$.

We now give an equivalent moduli problem. Write $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_{\ell} \subset \mathbb{A}_f^p$ and for any $\mathcal{O}_E \left[\frac{1}{p} \right]$ -lattice $L \subset V$,

write $\widehat{L}^{(p)} = L \otimes \widehat{\mathbb{Z}}^{(p)} \subset V(\mathbb{A}_f^p)$. We can find a $\mathcal{O}_E \left[\frac{1}{p} \right]$ -lattice $L \subset V$ satisfying the conditions:

$$(2.1) \quad \begin{aligned} K^p &\subset \left\{ g \in G(\mathbb{A}_f^p), g(\widehat{L}^{(p)}) = \widehat{L}^{(p)} \right\} \\ \varphi(L, L) &\subset \mathbb{Z} \left[\frac{1}{p} \right] \end{aligned}$$

(see notations for the definition of φ). The determinant of $\varphi : L \times L \rightarrow \mathbb{Z} \left[\frac{1}{p} \right]$ is a square in $\mathbb{Z} \left[\frac{1}{p} \right]$ and is well defined up to an invertible element. Let $d \in \mathbb{Z}$ coprime to p such that $\det(\varphi) = d^2$. We consider the moduli problem \mathfrak{F} classifying the following data, up to isomorphism: For any noetherian \mathcal{O}_{E_p} -scheme S ,

- (1) An abelian scheme A of dimension n over S .
- (2) A polarization $\lambda : A \rightarrow A^\vee$ of degree d^2 .
- (3) An action $\iota : \mathcal{O}_E \hookrightarrow \text{End}(A)$ compatible with λ .
- (4) A $\pi_1(S, s)$ -stable K^p -orbit of compatible isomorphisms $\overline{\eta} : \widehat{L}^{(p)} \xrightarrow{\sim} T_f^p(A_s) = \prod_{\ell \neq p} T_\ell(A_s)$ such that the following diagram commutes

$$\begin{array}{ccccc} \widehat{L}^{(p)} & \times & \widehat{L}^{(p)} & \longrightarrow & \widehat{\mathbb{Z}}^{(p)} \\ \eta^p \downarrow & & \eta^p \downarrow & & \downarrow \theta \\ T_f^p(A_s) & \times & T_f^p(A_s) & \longrightarrow & \widehat{\mathbb{Z}}^{(p)}(1) \end{array}$$

where θ is some $\widehat{\mathbb{Z}}^{(p)}$ -linear isomorphism. Further, $(A, \iota, \lambda, \overline{\eta})$ satisfies the determinant condition.

Proposition 8 below is well known, we will skip the proof.

Proposition 8. The natural map $\mathfrak{F} \rightarrow Sh_K$ is an isomorphism of functors.

Remark 9. For K^p small enough, every automorphism of a tuple $(A, \iota, \lambda, \overline{\eta})$ is trivial.

2.2. Dieudonné modules. We write $\overline{Sh_K} = Sh_K \times \kappa(\mathcal{O}_{E_p})$ for the special fibre of Sh_K . It is equidimensional of dimension $n - 1$. In order to study $\overline{Sh_K}$, we use (covariant) Dieudonné theory.

2.2.1. Some definitions. Let k' be an algebraically closed field containing $\kappa(\mathcal{O}_{E_p})$ and let $W = W(k')$ be the ring of Witt vectors and $W_{\mathbb{Q}} = W \otimes \mathbb{Q}$. The choice of k induces an embedding $\varrho : E_p \hookrightarrow W_{\mathbb{Q}}$. A *Dieudonné module* is a free W -module M of finite rank together with a σ -linear endomorphism F and a σ^{-1} -endomorphism V of M such that $FV = VF = p$.

A *Dieudonné space* over k is a finite-dimensional k -vector space together with a Frob_k -linear endomorphism and a Frob_k^{-1} -endomorphism V of M such that $FV = VF = 0$. If M is a Dieudonné module, then $\overline{M} = \frac{M}{pM}$ is a Dieudonné space which satisfies

$$(2.2) \quad \text{Im}(F) = \text{Ker}(V) \quad \text{and} \quad \text{Im}(V) = \text{Ker}(F).$$

An \mathcal{O}_{E_p} -*Dieudonné module* over k is a Dieudonné module endowed with a W -linear \mathcal{O}_{E_p} -action commuting with F, V . We define similarly the notion of \mathcal{O}_{E_p} -*Dieudonné space*. The \mathcal{O}_{E_p} -action induces a decomposition $M = M_e \oplus M_{\overline{e}}$ where M_e (resp. $M_{\overline{e}}$) is the submodule where \mathcal{O}_{E_p} acts via ϱ (resp. $\overline{\varrho}$). We define the *signature* of an \mathcal{O}_{E_p} -Dieudonné module to be the pair

$$\left(\dim_k \left(\frac{M_e}{VM_{\overline{e}}} \right), \dim_k \left(\frac{M_{\overline{e}}}{VM_e} \right) \right).$$

If A is an abelian variety over k and $M = \mathbb{D}(A)$, then $\text{Lie}(A) = \frac{M}{VM}$. We define in a similar fashion the signature of an \mathcal{O}_{E_p} -Dieudonné space. The signatures of M and \overline{M} are the same.

A *quasi-unitary Dieudonné module* over k is an \mathcal{O}_{E_p} -Dieudonné module endowed with a non-degenerate alternating pairing $\langle, \rangle : M \times M \rightarrow W_{\mathbb{Q}}$ such that $\forall e \in \mathcal{O}_{E_p}, \forall x, y \in M, \langle ex, y \rangle = \langle x, \overline{e}y \rangle$ and $\langle Fx, y \rangle = \sigma \langle x, Vy \rangle$. We call M *unitary* if $\langle, \rangle : M \times M \rightarrow W$ is perfect. We define similarly the notion of *unitary Dieudonné space*.

A *unitary isocrystal* over k is a finite-dimensional $W_{\mathbb{Q}}$ -vector space N together with endomorphisms F, V , an \mathcal{O}_{E_p} -action, a $W_{\mathbb{Q}}$ -bilinear pairing $\langle, \rangle : N \times N \rightarrow W_{\mathbb{Q}}$ subject to the same hypotheses as above. If M is a quasi-unitary Dieudonné module, then $M \otimes W_{\mathbb{Q}}$ is a unitary isocrystal. If $\lambda \in \mathbb{Q}$, we denote by N_{λ} “the” simple isocrystal of slope λ . We say that an isocrystal is supersingular if all its slopes are $\frac{1}{2}$.

2.2.2. *Dieudonné theory.* Dieudonné theory gives an equivalence of categories between unitary Dieudonné modules over k and p -divisible groups over k (with polarization and \mathcal{O}_{E_p} -action). For a definition of these objects, see [Büttel and Wedhorn(2006)] section 2. Similarly, there is an equivalence of categories between unitary Dieudonné spaces over k which satisfy (2.2) and truncated Barsotti-Tate groups of level 1 (or BT_1) over k (with polarization and \mathcal{O}_{E_p} -action). See [Grothendieck(1974)] definition 3.2 for these statements.

2.2.3. *Examples.*

- (1) Let SS be the following Dieudonné module: It has a W -basis (g, h) such that $SS_e = Wg$, $SS_{\bar{e}} = Wh$, the endomorphisms F, V are defined by $F(g) = h = -V(g)$, and the pairing is given by $\langle g, h \rangle = 1$. This is a unitary Dieudonné module of signature $(1, 0)$ and slope $\frac{1}{2}$.
- (2) Let $d \geq 1$ be an integer. Define a unitary Dieudonné module $\mathbb{B}(d)$ as follows: It has a W -basis (e_i, f_i) , $i \in \{1, \dots, d\}$ with $e_i \in \mathbb{B}(d)_e$ and $f_i \in \mathbb{B}(d)_{\bar{e}}$. The endomorphisms F, V are given by

$$\begin{aligned} F(f_1) &= (-1)^d e_n \\ F(e_i) &= f_{i-1} \text{ for } i = 2, \dots, d \\ V(f_d) &= e_1 \\ V(e_i) &= f_{i+1} \text{ for } i = 1, \dots, d-1. \end{aligned}$$

The alternating form is defined by $\langle e_i, f_j \rangle = (-1)^{i-1} \delta_{i,j}$. This is a unitary Dieudonné module of signature $(d-1, 1)$. If d is odd, every slope of $\mathbb{B}(d) \otimes W_{\mathbb{Q}}$ is $\frac{1}{2}$. If d is even, its slopes are $\frac{1}{2} \pm \frac{1}{d}$ (cf. [Büttel and Wedhorn(2006)] lemma 3.3).

2.2.4. *Classification.* We classify below isocrystals and Dieudonné spaces which come into play in our situation. We refer to [Büttel and Wedhorn(2006)], section 3.1 and 3.6 respectively for the proofs.

Proposition 10. *Let M be a unitary Dieudonné module of signature $(n-1, 1)$ and N its isocrystal. Then*

$$N \simeq N(r) \times (N_{\frac{1}{2}})^{n-2r}$$

where r is an integer $0 \leq r \leq \frac{n-1}{2}$ and:

$$N(r) = \begin{cases} 0 & \text{if } r = 0 \\ N_{\frac{1}{2}-\frac{1}{2r}} \oplus N_{\frac{1}{2}+\frac{1}{2r}} & \text{if } r > 0 \text{ is even} \\ N_{\frac{1}{2}-\frac{1}{2r}}^2 \oplus N_{\frac{1}{2}+\frac{1}{2r}}^2 & \text{if } r \text{ is odd.} \end{cases}$$

Proposition 11. *Let \overline{M} be a unitary Dieudonné space of signature $(n-1, 1)$. There is an integer $1 \leq r \leq n$ such that \overline{M} is isomorphic to*

$$\overline{\mathbb{B}}(r) \oplus \overline{SS}^{n-r}.$$

2.3. Stratifications.

2.3.1. *Ekedahl-Oort stratification.* Applying proposition 11 to M gives us an integer $1 \leq r \leq n$. This defines a stratification

$$\overline{Sh}_K = \bigsqcup_{r=1}^n \mathcal{M}_r$$

where \mathcal{M}_r is the locus where \overline{M} is isomorphic to $\overline{\mathbb{B}}(r) \oplus \overline{SS}^{n-r}$. A point in \mathcal{M}_r and its Dieudonné module are said of type r . All the \mathcal{M}_r are equidimensional and the dimensions are given by:

$$\begin{aligned} \dim(\mathcal{M}_{2i}) &= n - i \\ \dim(\mathcal{M}_{2i+1}) &= i \end{aligned}$$

(cf. *ibid*, 5.4).

2.3.2. *Newton polygon stratification.* The Newton polygon stratification is given by isomorphism classes of unitary isocrystals. It happens to be coarser than the Ekedahl-Oort stratification. It reads:

$$\overline{Sh}_K = \mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \dots \sqcup \mathcal{M}_{n-1} \sqcup \bigsqcup_{r \text{ odd}} \mathcal{M}_r.$$

The stratum \mathcal{M}_{2r} is also the locus where the unitary isocrystal is isomorphic to $N(r) \times (N_{\frac{1}{2}})^{n-2r}$. The supersingular locus is $\overline{Sh}_K^{ss} = \bigsqcup_{r \text{ odd}} \mathcal{M}_r$ and has dimension $\frac{n-1}{2}$ (*ibid*, proposition 5.5). Finally, we state a result on the geometric structure of \overline{Sh}_K^{ss} . For the proof, see [Volgaard and Wedhorn(2011)], theorem 5.2.

Theorem 12. *For K^p sufficiently small, the supersingular locus \overline{Sh}_K^{ss} is equidimensional of dimension $\frac{n-1}{2}$ and locally of complete intersection. Its smooth locus is the open Ekedahl-Oort stratum \mathcal{M}_n .*

3. MODULI SPACE OF p -ISOGENIES

3.1. **The moduli problem.** We define a moduli space classifying p -isogenies. Let S be an \mathcal{O}_{E_p} -scheme and $\underline{A}_i = (A_i, \iota_i, \overline{\lambda}_i, \overline{\eta}_i)$, $i \in \{1, 2\}$ two tuples corresponding to S -valued points of Sh_K . A p -isogeny $f : \underline{A}_1 \rightarrow \underline{A}_2$ is an \mathcal{O}_{E_p} -linear isogeny such that $p^c \lambda_1 = f^\vee \circ \lambda_0 \circ f$ for some $c \geq 0$, which we call the multiplier. This implies $\deg(f) = p^{cn}$.

Let $p\text{-}\mathcal{I}so$ be the \mathcal{O}_{E_p} -scheme classifying p -isogenies. Two p -isogenies $f : \underline{A}_1 \rightarrow \underline{A}_2$ and $f' : \underline{A}'_1 \rightarrow \underline{A}'_2$ are identified if there are prime-to- p isogenies $h_i : \underline{A}_i \rightarrow \underline{A}'_i$ for $i \in \{1, 2\}$ such that $f' \circ h_1 = h_2 \circ f$. The p -isogenies of multiplier c form an open and closed subscheme $p\text{-}\mathcal{I}so^{(c)} \subset p\text{-}\mathcal{I}so$.

Let S be an \mathcal{O}_{E_p} -scheme. Write $\mathfrak{I}(S)$ for the moduli problem classifying p -isogenies between points of $\mathfrak{F}(S)$ (see 2.1 for the definition of \mathfrak{F}), up to isomorphisms. The natural map $\mathfrak{I} \rightarrow p\text{-}\mathcal{I}so$ is an isomorphism of functors.

Let $s, t : p\text{-}\mathcal{I}so \rightarrow Sh_K$ be the maps sending an isogeny to its source and target, respectively. The restrictions $s, t : p\text{-}\mathcal{I}so^{(c)} \rightarrow Sh_K$ are proper for $c \geq 0$ ([Moonen(2004)], 4.2.1).

The “multiplication by p ” map sends \underline{A} to $p : \underline{A} \rightarrow \langle p \rangle \underline{A}$ (for $q \in p^{\mathbb{Z}}, \langle q \rangle$ multiplies the level structure by q). This defines a section of s . As s is separated, its image is a reduced closed subscheme $\langle p \rangle \subset p\text{-}\mathcal{I}so^{(2)}$.

The ordinary locus $p\text{-}\mathcal{I}so^{ord} \times \kappa(\mathcal{O}_{E_p})$ is defined as the inverse image of \overline{Sh}_K^{ord} by s (or t). We define similarly the supersingular locus $p\text{-}\mathcal{I}so^{ss} \times \kappa(\mathcal{O}_{E_p})$. On the special fibre, there is a Frobenius section of s . It sends a tuple \underline{A} to the Frobenius isogeny $F_{\underline{A}} : \underline{A} \rightarrow \underline{A}^{(p^2)}$. The level structure on $\overline{\eta}^{(p^2)}$ on $A^{(p^2)}$ is compatible with $\overline{\eta}$ through F_A . The image of s is a reduced closed subscheme $F \subset p\text{-}\mathcal{I}so \times \kappa(\mathcal{O}_{E_p})$, which is a union of irreducible components of $p\text{-}\mathcal{I}so \times \kappa(\mathcal{O}_{E_p})$, because s is finite and flat over the ordinary locus ([Moonen(2004)], 4.2.2) and \overline{Sh}_K^{ord} is dense in \overline{Sh}_K . This follows also from the fact that $p\text{-}\mathcal{I}so \times \kappa(\mathcal{O}_{E_p})$ is equidimensional of dimension $n-1$, as we will show later. By duality, we also have the Verschiebung map $V_A : A^{(p^2)} \rightarrow A$. Notice that $V_A \circ F_A = p^2$ so taking into account level structures, the Verschiebung is a map $V_{\underline{A}} : \langle p^{-2} \rangle \underline{A}^{(p^2)} \rightarrow \underline{A}$.

3.2. **The \mathbb{Q} -algebra $\mathbb{Q}[p\text{-}\mathcal{I}so \times L]$.** Composition of isogenies defines a morphism

$$c : p\text{-}\mathcal{I}so \times_{t,s} p\text{-}\mathcal{I}so \longrightarrow p\text{-}\mathcal{I}so$$

which is proper (cf. [Moonen(2004)], 4.2.1). Let L be a field and $\mathcal{O}_{E_p} \rightarrow L$ a homomorphism. Let $Z_{\mathbb{Q}}(p\text{-}\mathcal{I}so \times L)$ denote the group of algebraic cycles of $p\text{-}\mathcal{I}so \times L$, with \mathbb{Q} -coefficients. For cycles Y_1, Y_2 , we define:

$$Y_1 \cdot Y_2 = c_*(Y_1 \times_{t,s} Y_2).$$

Extending this product bilinearly, we get a ring structure on $Z_{\mathbb{Q}}(p\text{-}\mathcal{I}so \times L)$, with identity $p\text{-}\mathcal{I}so^{(0)} \times L$. Let $\mathbb{Q}[p\text{-}\mathcal{I}so \times L]$ be the \mathbb{Q} -subalgebra generated by the irreducible components.

Define the \mathbb{Q} -algebra $\mathbb{Q}[p\text{-}\mathcal{I}so^{ord} \times \kappa(\mathcal{O}_{E_p})]$ in a similar fashion as hereabove. We may view F as an element of $\mathbb{Q}[p\text{-}\mathcal{I}so^{ord} \times \kappa(\mathcal{O}_{E_p})]$ or $\mathbb{Q}[p\text{-}\mathcal{I}so \times \kappa(\mathcal{O}_{E_p})]$.

3.3. A commutative diagram. Let $\mathcal{H}_0(G(\mathbb{Q}_p)//K_p) \subset \mathcal{H}(G(\mathbb{Q}_p)//K_p)$ be the subalgebra of \mathbb{Q} -valued functions that have support contained in $G(\mathbb{Q}_p) \cap \text{End}(\Lambda)$. There is a \mathbb{Q} -algebra homomorphism

$$h : \mathcal{H}_0(G(\mathbb{Q}_p)//K_p) \longrightarrow \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times E_p]$$

which we will explain briefly. Let L be a field containing E_p and $f : \underline{A}_1 \rightarrow \underline{A}_2$ corresponding to an L -valued point in $p\text{-}\mathcal{I} \text{ sog} \times E_p$. Choose isomorphisms $\alpha_i : \Lambda \simeq T_p(A_i), i \in \{0, 1\}$. Then $\alpha_2^{-1} \circ V_p f \circ \alpha_1 : \Lambda \otimes \mathbb{Q}_p \rightarrow \Lambda \otimes \mathbb{Q}_p$ is an element of $G(\mathbb{Q}_p) \cap \text{End}(\Lambda)$. Its class $\tau(f)$ in $K_p \backslash G(\mathbb{Q}_p)/K_p$ is independent of the choices involved. The function τ is constant on irreducible components of $p\text{-}\mathcal{I} \text{ sog} \times E_p$. Then h maps $1_{K_p g K_p}$ to the sum of irreducible components $C \subset p\text{-}\mathcal{I} \text{ sog} \times E_p$ such that $\tau(C) = K_p g K_p$. The specialization map

$$\sigma : \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times E_p] \longrightarrow \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})]$$

is defined as follows: Let C an irreducible composant of $p\text{-}\mathcal{I} \text{ sog} \times E_p$ and \mathcal{C} the scheme-theoretic image of C by the open immersion $p\text{-}\mathcal{I} \text{ sog} \times E_p \hookrightarrow p\text{-}\mathcal{I} \text{ sog}$. Then $\sigma(C) = [\mathcal{C} \times \kappa(\mathcal{O}_{E_p})]$.

The cocharacter μ is defined over E . Let $M \subset G_{\mathbb{Q}_p}$ be the centralizer of $\mu\bar{\mu}$. We denote again by $\mathcal{H}_0(M(\mathbb{Q}_p)//(K_p \cap M(\mathbb{Q}_p)))$ the functions with support in Λ . We have a commutative diagram of \mathbb{Q} -algebra homomorphisms

$$(3.1) \quad \begin{array}{ccc} \mathcal{H}_0(G(\mathbb{Q}_p)//K_p) & \xrightarrow{h} & \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times E_p] \\ \downarrow \dot{S}_M^G & & \downarrow \sigma \\ & & \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})] \\ & & \uparrow \text{cl} \quad \downarrow \text{ord} \\ \mathcal{H}_0(M(\mathbb{Q}_p)//(K_p \cap M(\mathbb{Q}_p))) & \xrightarrow{h_0} & \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog}^{\text{ord}} \times \kappa(\mathcal{O}_{E_p})] \end{array}$$

The morphism \dot{S} is the twisted Satake homomorphism (see [Wedhorn(2000)] §1). The map ord is defined by intersecting with the ordinary locus, and the map cl is defined by taking the closure of a cycle of $p\text{-}\mathcal{I} \text{ sog}^{\text{ord}} \times \kappa(\mathcal{O}_{E_p})$. The big square is commutative. In this context, the ‘‘congruence relation’’ means the following conjecture:

Conjecture. Consider the polynomial H_p inside $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})]$ via the morphism $\sigma \circ h^0$. The element F lies in the center of this ring and the following relation holds:

$$H_p(F) = 0.$$

This relation makes sense since F belongs to the center of $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})]$, as we shall see in section 5. We mention the following theorem, due to B. Moonen in [Moonen(2004)].

Theorem. Consider the polynomial H_p inside $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog}^{\text{ord}} \times \kappa(\mathcal{O}_{E_p})]$ via the morphism $\text{ord} \circ \sigma \circ h^0$. In this ring, the following relation holds:

$$H_p(F) = 0.$$

4. THE SUPERSINGULAR LOCUS

4.1. The moduli space \mathcal{N}' . Uniformization theory from [Rapoport and Zink(1996)] can be used in order to study the supersingular locus of \overline{Sh}_K . In [Vollaard(2010), Vollaard and Wedhorn(2011)], the authors give the geometric structure of $\overline{Sh}_K^{\text{ss}}$. We state below their main results.

Let $K^p \subset G(\mathbb{A}_f^p)$ be an open compact subgroup. We fix a tuple $\underline{A}' = (A', \iota', \bar{\lambda}', \bar{\eta}')$ over $\bar{\mathbb{F}}$. Using the same conventions as in [Rapoport and Zink(1996)], $\bar{\eta}'$ is a K^p -orbit of isomorphisms: $\bar{\eta}' : H_1(A', \mathbb{A}_f^p) \rightarrow V(\mathbb{A}_f^p)$. We assume that A' is supersingular. We denote by \underline{X}' its p -divisible group over $\bar{\mathbb{F}}$ and we write $M' = \mathbb{D}(A')$ and $N' = M' \otimes W_{\mathbb{Q}}$.

The formal scheme \mathcal{N}' over $\bar{\mathbb{F}}$ classifies the following pairs (\underline{X}, ρ_X) up to prime-to- p -isogenies. Given a $\bar{\mathbb{F}}$ -scheme S , \underline{X} is a p -divisible group with unitary structure of signature $(n-1, 1)$ over S and $\rho_X : \underline{X} \rightarrow \underline{X}'_S$ is a quasi-isogeny such that $\rho_X^*(\bar{\lambda}') = p^c \bar{\lambda}$ for some $c \in \mathbb{Z}$.

Dieudonné theory gives a bijection between $\mathcal{N}'(\overline{\mathbb{F}})$ and the set of quasi-unitary Dieudonné modules $M \subset N'$ of signature $(n-1, 1)$ such that $p^c M^\vee = M$ for some $c \in \mathbb{Z}$. If $(\underline{X}, \rho_X) \in \mathcal{N}'(S)$, there is a unique tuple $\underline{A} = (A, \iota, \overline{\lambda}, \overline{\eta})$ over S with a quasi-isogeny $f : \underline{A} \rightarrow \underline{A}'$ lifting ρ_X . We write $\underline{A} = \rho_X^* \underline{A}'$.

If $g \in G(\mathbb{A}_f)$, and $\underline{A} = (A, \iota, \overline{\lambda}, \overline{\eta})$ is a tuple over S , we define $\langle g \rangle \underline{A} = (A, \iota, \overline{\lambda}, \overline{g \circ \eta})$. The uniformization morphism is given by:

$$\begin{aligned} \Theta : \mathcal{N}' \times G(\mathbb{A}_f^p) &\longrightarrow \overline{Sh_K}^{ss} \times \overline{\mathbb{F}} \\ (X, \rho_X) \times g &\longmapsto \langle g \rangle \rho_X^* \underline{A}' \end{aligned}$$

Let I be the algebraic group over \mathbb{Q} of $\mathcal{O}_{E,(p)}$ -linear quasi-isogenies in $\text{End}^0(A')$ compatible with $\overline{\lambda}'$. We have a natural homomorphism $\alpha_p : I(\mathbb{Q}_p) \hookrightarrow J(\mathbb{Q}_p)$, where J denotes the \mathbb{Q}_p -algebraic group of automorphisms of N' respecting the polarization up to factor. An element $\eta' \in \overline{\eta}'$ provides a homomorphism $\alpha^p : I(\mathbb{Q}) \rightarrow J(\mathbb{A}_f^p)$ (for more details, see [Rapoport and Zink(1996)], 6.15). We have the following theorem ([Rapoport and Zink(1996)] theorem 6.30):

Theorem 13. *The uniformization theorem induces an isomorphism of $\overline{\mathbb{F}}$ -schemes:*

$$I(\mathbb{Q}) \backslash \mathcal{N}'_{red} \times G(\mathbb{A}_f^p) / K^p \longrightarrow \overline{Sh_K}^{ss} \times \overline{\mathbb{F}}.$$

Write $I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p) / K^p = \{g_1, \dots, g_m\}$ and $\Gamma_j = I(\mathbb{Q}) \cap g_j K^p g_j^{-1}$. There is a decomposition:

$$I(\mathbb{Q}) \backslash \mathcal{N}'_{red} \times G(\mathbb{A}_f^p) / K^p = \coprod_{j=1}^m \Gamma_j \backslash \mathcal{N}'_{red}.$$

We now recall some results from [Volgaard(2010), Volgaard and Wedhorn(2011)]. Notice that in these articles the signature of A' is $(1, n-1)$. That's why we modify slightly the definition of $\mathcal{L}_i(n)$ (the integer i is replaced by $i-1$). The scheme \mathcal{N}'_{red} has a stratification

$$\mathcal{N}'_{red} = \bigcup_{i \in 2\mathbb{Z}} \mathcal{N}'_{red,i}$$

where $\mathcal{N}'_{red,i}$ is the open and closed subscheme of elements of multiplier i . Observe that $\mathcal{N}'_{red,i}$ is empty if i is odd ([Volgaard and Wedhorn(2011)] (1.5.1)). For i even, all the $\mathcal{N}'_{red,i}$ are isomorphic to one another (*ibid*, proposition 1.1). Write $\mathbf{N}'_0 = \{x \in N'_e, \tau x = x\}$, where $\tau = p^{-1}F^2$. This is a \mathbb{Q}_{p^2} -hermitian space for the form $\{x, y\} = \alpha \langle x, Fy \rangle$. Define:

$$(4.1) \quad \mathcal{L}_i(n) = \{L \subset \mathbf{N}'_0, \mathbb{Z}_{p^2}\text{-lattice}, L = p^{i-1}L^\wedge\}$$

where L^\wedge is the dual lattice with respect to $\{, \}$. For each $L \in \mathcal{L}_i(n)$, there is an associated closed subscheme $\mathcal{N}'_L \subset \mathcal{N}'_{red,i}$. We have the following decomposition in irreducible components:

$$\mathcal{N}'_{red,i} = \bigcup_{L \in \mathcal{L}_i(n)} \mathcal{N}'_L$$

(*ibid*, theorem 4.2). The \mathcal{N}'_L are all isomorphic for $L \in \mathcal{L}_i(n)$, smooth, of dimension $\frac{n-1}{2}$. We say that a point $(\underline{X}, \rho_X) \in \mathcal{N}'_{red}$ has type r if its Dieudonné module has type r (see 2.4.1). The smooth locus of \mathcal{N}'_{red} is the set of points of type n .

Further, there is a bijection between quasi-unitary superspecial Dieudonné modules $M \subset N'$ of signature $(n, 0)$ such that $p^{i-1}M^\vee = M$ and lattices in $\mathcal{L}_i(n)$. The bijection is given by $M \mapsto M_e^\tau$. If $L \in \mathcal{L}_i(n)$, write L^+ for the associated superspecial Dieudonné module of signature $(n, 0)$. We thus get a bijection between irreducible components of $\mathcal{N}'_{red,i}$ and quasi-unitary superspecial Dieudonné modules of signature $(n, 0)$ which satisfy $p^{i-1}M^\vee = M$. If y a point in $\mathcal{N}'_{red,i}(\overline{\mathbb{F}})$ with Dieudonné module M , then y lies in $\mathcal{N}'_L(\overline{\mathbb{F}})$ if and only if $M \subset L^+$ (*ibid*, lemma 3.3). If $y \in \mathcal{N}'_L(\overline{\mathbb{F}})$ has type n , then $L^+ = \Lambda^+(M)$, the smaller superspecial Dieudonné module containing M .

4.2. Dimension of the fibers of s, t . Let $c \geq 0$ be a fixed even integer and x be an $\overline{\mathbb{F}}$ -valued point of $\overline{Sh_K}^{ss}$, corresponding to a tuple $\underline{A}' = (A', \iota', \overline{\lambda}, \overline{\eta})$ over $\overline{\mathbb{F}}$. Let M' be its Dieudonné module and N' its isocrystal. We write $t_c^{-1}(x)$ for the fibre of t above x in $p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}}$. We consider the moduli space \mathcal{N}'_{red} associated to \underline{A}' , as above. We assume that K^p satisfies the condition of remark 9. Then, there is a well-defined morphism of $\overline{\mathbb{F}}$ -schemes:

$$\epsilon : t_c^{-1}(x) \longrightarrow \mathcal{N}'_{red}$$

sending an isogeny $f : \underline{A} \rightarrow \underline{A}'$ on the induced isogeny $f : \underline{X} \rightarrow \underline{X}'$ on the p -divisibles groups (forgetting the level structure). It can be showed that ϵ is proper using the valuative criterion. Further, ϵ is injective on S -points for all $\overline{\mathbb{F}}$ -scheme S , since \underline{A} can be reconstructed from $f : \underline{X} \rightarrow \underline{X}'$. Thus ϵ is a closed immersion ([Grothendieck(1966)], 8.11.5).

The $\overline{\mathbb{F}}$ -points of $t_c^{-1}(x)$ are in bijection with the quasi-unitary Dieudonné M modules over k of signature $(n-1, 1)$ satisfying $M \subset M'$ and $p^c M^\vee = M$. The map ϵ induces the natural injection of this set into $\mathcal{N}'_{red}(\overline{\mathbb{F}})$. If $f : \underline{A} \rightarrow \underline{A}'$ lies in $t_c^{-1}(x)$, then $\rho_X^* \underline{A}' = \underline{A}$. Embed \mathcal{N}'_{red} into $\mathcal{N}'_{red} \times G(\mathbb{A}_f)$ by $\alpha : z \mapsto (z, 1)$. There is a commutative diagram:

$$\begin{array}{ccc} I(\mathbb{Q}) \backslash \mathcal{N}'_{red} \times G(\mathbb{A}_f^p) / K^p & \xrightarrow{\Theta} & \overline{Sh_K}^{ss} \times k \\ \alpha \uparrow & & \uparrow s \\ \mathcal{N}'_{red} & \xleftarrow{\epsilon} & t_c^{-1}(x) \end{array}$$

Proposition 14. *The restriction of s to $t_c^{-1}(x)$ is a finite morphism.*

Proof. The restriction of α to any quasi-compact subscheme of \mathcal{N}'_{red} is finite (see [Volleard and Wedhorn(2011)], 5.4). \square

Corollary 15. *The morphism*

$$p\text{-}\mathcal{I} \text{ sog}^{(c), ss} \times \kappa(\mathcal{O}_{E_p}) \xrightarrow{(s, t)} \overline{Sh_K}^{ss} \times \overline{Sh_K}^{ss}$$

is finite.

Proof. It is proper and quasi-finite. \square

Corollary 16. *Let $x \in \overline{Sh_K}^{ss}(\overline{\mathbb{F}})$ and $c \geq 2$ be an even integer. The dimension of the fibre $t_c^{-1}(x)$ is $\frac{n-1}{2}$.*

Proof. Clearly $\dim(t_c^{-1}(x)) \leq \frac{n-1}{2}$. Let M be the Dieudonné module associated to x . Let M_0 be any quasi-unitary Dieudonné module of signature $(n, 0)$ such that $M_0 \subset M$ and $M_0^\vee = p^{c-1} M_0$. The irreducible component of \mathcal{N}'_{red} associated to M_0 is then contained in $t_c^{-1}(x)$. \square

Remark 17. When $c = 2$ and x has type n , there is only one such M_0 (namely $p\Lambda^+(M)$). Therefore, the fibre $t_c^{-1}(x)$ has only one irreducible component of dimension $\frac{n-1}{2}$.

4.3. Irreducible components of $p\text{-}\mathcal{I} \text{ sog} \times \overline{\mathbb{F}}$.

Proposition 18. *$p\text{-}\mathcal{I} \text{ sog} \times \overline{\mathbb{F}}$ is equidimensional of dimension $n-1$. If an irreducible component of $p\text{-}\mathcal{I} \text{ sog} \times \overline{\mathbb{F}}$ intersects the ordinary locus, it is contained in the closure of $p\text{-}\mathcal{I} \text{ sog}^{ord} \times \overline{\mathbb{F}}$. Otherwise, all its points are supersingular.*

Proof. Let C be an irreducible component of $p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}}$ for $c \geq 0$. Using [Büttel and Wedhorn(2006)] (proposition 6.15), we have $\dim(C) \geq n-1$. Suppose that C intersects $p\text{-}\mathcal{I} \text{ sog}^{(c), ord} \times \overline{\mathbb{F}}$. Since the ordinary locus is open, C is contained in its closure and $\dim(C) = n-1$. Suppose that C has no ordinary point, and that there exists a non-supersingular point $z \in C$. There is an open subset $U \subset \overline{Sh_K}$ containing $t(z)$ such that $U \cap \overline{Sh_K}^{ss} = \emptyset$. Then z is in $t^{-1}(U) \cap C$ so this is a non empty dense open subset of C . By [Büttel and Wedhorn(2006)] corollary 7.3, the map t is finite over $t^{-1}(U)$, thus:

$$\dim(t^{-1}(U) \cap C) = \dim(t(t^{-1}(U) \cap C)) \leq \dim(t(C)) < n-1$$

because $t(C)$ doesn't meet $\overline{Sh_K}^{ord}$. This contradicts $\dim(C) \geq n-1$. We have shown that a component not intersecting $p\text{-}\mathcal{J} \text{ sog}^{(c),ord} \times \overline{\mathbb{F}}$ is supersingular. Finally, let C be a supersingular irreducible component. We have $\dim(\overline{Sh_K}^{ss}) = \frac{n-1}{2} = \dim(t_c^{-1}(x))$ for all $x \in \overline{Sh_K}^{ss}(\overline{\mathbb{F}})$, so $\dim(C) = n-1$. \square

4.4. The map (s, t) .

Theorem 19. *Let $c \geq 0$ be an even integer. There exists $K^p \subset G(\mathbb{A}_f^p)$ such that*

$$p\text{-}\mathcal{J} \text{ sog}_K^{(c),ss} \times \kappa(\mathcal{O}_{E_p}) \xrightarrow{(s,t)} \overline{Sh_K}^{ss} \times \overline{Sh_K}^{ss}$$

is a closed immersion.

Proof. We will use the moduli problems $\mathfrak{F}, \mathfrak{J}$ described in 2.1 and 3.1. Choose an $\mathcal{O}_E \left[\frac{1}{p} \right]$ -lattice $L \subset V$ satisfying the condition (2.1). Let $x_i = (A_i, \lambda_i, \iota_i, \overline{\eta}_i)$, $i \in \{0, 1\}$ be two points of $\mathfrak{F}(\overline{\mathbb{F}})$ with multiplier c . We assume that there is an isogeny $h : (A_0, \lambda_0, \iota_0) \rightarrow (A_1, \lambda_1, \iota_1)$. We write $R = \text{Hom}(A_0, A_1)$ for the group of homomorphisms (with no compatibility condition). If $f, g \in R$, we write

$$\langle f, g \rangle = \text{Tr} \left(d^2 \lambda_0^{-1} \circ f^\vee \circ \lambda_1 \circ g \right)$$

where $\text{Tr} : \text{End}(A_0) \rightarrow \mathbb{Z}$ is the trace morphism. Since λ_0 has degree d^2 , the quasi-isogeny $d^2 \lambda_0^{-1}$ is an isogeny, thus $\langle f, g \rangle \in \mathbb{Z}$. The form $\langle \cdot, \cdot \rangle$ is symmetric, positive definite. Indeed, h identifies this form with one on $\text{End}(A_0) \otimes \mathbb{Q}$ which is positive definite because the Rosati involution is. Define $q(f) = \langle f, f \rangle$ for $f \in R$. For a p -isogeny f of multiplier c , we have

$$(4.2) \quad q(f) = d^2 p^c.$$

Let $\ell \neq p$ be a prime. We have an injective morphism:

$$R \otimes \mathbb{Z}_\ell \hookrightarrow \text{Hom}_{\mathbb{Z}_\ell}(T_\ell(A_0), T_\ell(A_1)).$$

Each module $T_\ell(A_i)$ carries a pairing $\varepsilon^{\lambda_i} : T_\ell(A_i) \times T_\ell(A_i) \rightarrow \mathbb{Z}_\ell(1)$ associated to λ_i . If $f : V_\ell(A_0) \rightarrow V_\ell(A_1)$ is a \mathbb{Q}_ℓ -linear map, we define $f^* : V_\ell(A_1) \rightarrow V_\ell(A_0)$ by the formula:

$$\langle fx, y \rangle^{\lambda_1} = \langle x, f^*y \rangle^{\lambda_0}$$

and we write $\widehat{q}(f) = d^2 \text{Tr}(f^*f)$. If $f \in R$, the properties of the Weil pairing show that $f^* = \lambda_0^{-1} \circ f^\vee \circ \lambda_1$. This implies $q(f) = \widehat{q}(f)$. The level structures $\overline{\eta}_0, \overline{\eta}_1$ identify $\text{Hom}_{\mathbb{Z}_\ell}(T_\ell(A_0), T_\ell(A_1))$ with $\text{End}_{\mathbb{Z}_\ell}(L \otimes \mathbb{Z}_\ell)$. Again, we define an involution $*$ of $\text{End}_{\mathbb{Q}}(V)$ by the formula:

$$\psi(fx, y) = \psi(x, f^*y), \quad \forall f \in \text{End}_{\mathbb{Q}}(V), \forall x, y \in V$$

and we write $q_0(f) = d^2 \text{Tr}(ff^*)$. We get a commutative diagram

$$\begin{array}{ccccc} R \otimes \mathbb{Z}_\ell & \hookrightarrow & \text{Hom}_{\mathbb{Z}_\ell}(T_\ell(A_0), T_\ell(A_1)) & \xrightarrow{\simeq} & \text{End}_{\mathbb{Z}_\ell}(L \otimes \mathbb{Z}_\ell) \\ & \searrow q \otimes \mathbb{Z}_\ell & \downarrow \widehat{q} & \swarrow q_0 \otimes \mathbb{Z}_\ell & \\ & & \mathbb{Z}_\ell & & \end{array}$$

From now on, we assume that A_0, A_1 are supersingular. Then we have an isomorphism

$$\text{Hom}(A_0, A_1) \otimes \mathbb{Z}_\ell \xrightarrow{\simeq} \text{Hom}_{\mathbb{Z}_\ell}(T_\ell(A_0), T_\ell(A_1)).$$

The diagram above shows $(R, q) \otimes \mathbb{Z}_\ell \simeq (\text{End}_{\mathbb{Z}}(L), q_0) \otimes \mathbb{Z}_\ell$ as \mathbb{Z}_ℓ -quadratic spaces. At p , we have an isomorphism:

$$(4.3) \quad \text{Hom}(A_0, A_1) \otimes \mathbb{Z}_p \simeq \text{Hom}_{W[F, V]}(\mathbb{D}(A_0), \mathbb{D}(A_1))$$

because A_0, A_1 are supersingular. The pairings $\langle \cdot, \cdot \rangle_i : \mathbb{D}(A_i) \times \mathbb{D}(A_i) \rightarrow W$ induce a transformation $f \mapsto f^* :$

$$\text{Hom}(\mathbb{D}(A_0) \otimes W_{\mathbb{Q}}, \mathbb{D}(A_1) \otimes W_{\mathbb{Q}}) \rightarrow \text{Hom}(\mathbb{D}(A_1) \otimes W_{\mathbb{Q}}, \mathbb{D}(A_0) \otimes W_{\mathbb{Q}})$$

Then $q \otimes \mathbb{Z}_p$ becomes via (4.3) the form $f \mapsto d^2 \text{Tr}(ff^*)$ on $\text{Hom}_{W[F, V]}(\mathbb{D}(A_0), \mathbb{D}(A_1))$. Since the pairings are perfect, the discriminant of $q \otimes \mathbb{Z}_p$ is a unit in \mathbb{Z}_p .

To every $x_0, x_1 \in \mathfrak{F}(\overline{\mathbb{F}})$ (such that there exists an isogeny $h : x_0 \rightarrow x_1$ in $\mathfrak{I}(\overline{\mathbb{F}})$), we have associated a positive definite \mathbb{Z} -quadratic space (R, q) , whose discriminant is independant of x_0, x_1, h . Let \mathcal{L} be the set of isomorphism classes of the spaces (R, q) . It is finite since there are only finitely many isomorphism classes of positive definite \mathbb{Z} -quadratic spaces of given discriminant. For every fixed $C \geq 0$, we thus can find $N \geq 0$ coprime to p such that:

$$\forall (R, q) \in \mathcal{L}, \forall u, v \in R, \begin{cases} u \equiv v [NR] \\ q(u) = q(v) = C \end{cases} \implies u = v.$$

Take N satisfying this condition for $C = d^2 p^c$. Define $K'^p = \left\{ g \in K^p, (g-1)(\widehat{L}^{(p)}) \subset N\widehat{L}^{(p)} \right\}$ and write $\mathfrak{I}_{K'^p}$ the moduli problem for this new level structure. Let f, g be two isogenies in $\mathfrak{I}_{K'^p}(\overline{\mathbb{F}})$ of multiplier c , such that $s(f) = s(g)$ and $t(f) = t(g)$, denoted respectively x_0 and x_1 . Then (4.2) shows that $q(f) = q(g) = C$. By definition of K'^p , we have $f = g$ on $A_0[N]$. There exists $h \in R$ such that $f - g = Nh$, thus $f = g$. This shows that (s, t) is injective on supersingular $\overline{\mathbb{F}}$ -points.

Write $R = \frac{\mathbb{F}[t]}{(t^2)}$. Let f, g be two supersingular points in $\mathfrak{I}_{K'^p}(R)$ such that $s(f) = s(g)$ and $t(f) = t(g)$. We denote by f_0, g_0 the reduced isogenies on $\overline{\mathbb{F}}$. We have $(s, t)(f_0) = (s, t)(g_0)$, thus $f_0 = g_0$, and we deduce $f = g$ by [Conrad(2004)], lemma 3.1. This shows that (s, t) is a closed immersion. \square

5. CONGRUENCE RELATION

5.1. A few lemmas.

Theorem 20. *Let X, Y be irreducible schemes of finite type over a field. Let $f : X \rightarrow Y$ be a dominant morphism. Then there is an open dense subset $U \subset Y$ such that for all $y \in U$, we have:*

$$\dim(f^{-1}(y)) = \dim(X) - \dim(Y).$$

Proof. We may assume that X, Y are reduced. Using [Grothendieck(1966)] théorème 6.9.1, there exists an open dense subset $U \subset Y$ such that $f : f^{-1}(U) \rightarrow U$ is flat. Then use lemme 13.1.1 and corollaire 14.2.4 of *ibid.* \square

Corollary 21. *Let X, Y be schemes of finite type over a field. Let $f : X \rightarrow Y$ be a dominant morphism and $r \geq 0$. Assume that for all $y \in Y$, the dimension of $f^{-1}(y)$ is r . Then $\dim(X) - \dim(Y) = r$.*

Proof. This is a simple exercise. \square

Lemma 22. *Let $C \subset p\text{-}\mathcal{I}\text{sog}^{(c)} \times \overline{\mathbb{F}}$ be a supersingular irreducible component. Then $C_s := s(C)$ and $C_t := t(C)$ are irreducible components of $\overline{Sh_K}^{ss} \times \overline{\mathbb{F}}$.*

Proof. They are irreducible closed subsets of dimension $\geq \frac{n-1}{2}$ since the fibres have dimension $\leq \frac{n-1}{2}$. But $\overline{Sh_K}^{ss}$ is equidimensional of dimension $\frac{n-1}{2}$ ([Vollaard and Wedhorn(2011)] theorem 5.2), so the result follows. \square

Proposition 23. *Let $C_1, C_2 \subset p\text{-}\mathcal{I}\text{sog}^{(c)} \times \overline{\mathbb{F}}$ be supersingular irreducible components. Assume that the map (s, t) is a closed immersion on $p\text{-}\mathcal{I}\text{sog}_K^{(c), ss}$. Assume further that $C_{1,s} = C_{2,s}$ and $C_{1,t} = C_{2,t}$. Then $C_1 = C_2$.*

Proof. The map (s, t) induces a closed immersion $C_1 \hookrightarrow C_{1,s} \times C_{1,t}$. Since $C_{1,s}$ and $C_{1,t}$ are irreducible components of $\overline{Sh_K}^{ss} \times \overline{\mathbb{F}}$, they are smooth of dimension $\frac{n-1}{2}$. The product $C_{1,s} \times C_{1,t}$ is thus irreducible of dimension $n-1$, so (s, t) defines an isomorphism $C_1 \xrightarrow{\sim} C_{1,s} \times C_{1,t}$. The same holds for C_2 , and we deduce the result. \square

5.2. The Frobenius action. Let \mathcal{F} be the Frobenius map on $\overline{Sh_K}$ and $p\text{-}\mathcal{I}\text{sog} \times \kappa(\mathcal{O}_{E_p})$. If C is a cycle, write $|C|$ for its support.

Proposition 24. *Let \tilde{C} be an irreducible component of $\overline{Sh_K}^{ss} \times \overline{\mathbb{F}}$. We have:*

$$\mathcal{F}(\tilde{C}) = \langle p \rangle \tilde{C}.$$

Proof. Let $x \in \overline{Sh_K^{ss}}(\overline{\mathbb{F}})$ be a point of type n whose image lies in \tilde{C} . Write $M = \mathbb{D}(x)$. Then $t_2^{-1}(x)$ has a unique irreducible component C of dimension $\frac{n-1}{2}$ (see remark 17). More precisely, points $y \in p\text{-}\mathcal{I} \text{ sog}^{(2)}(\overline{\mathbb{F}})$ whose image lie in C correspond to quasi-unitary Dieudonné modules M' of signature $(n-1, 1)$ satisfying $p^2 M'^\vee = M'$ and $M' \subset p\Lambda^+(M)$. Clearly the isogenies $p : \langle p^{-1} \rangle x \rightarrow x$ and $V : \langle p^{-2} \rangle Fx \rightarrow x$ belong to $t_2^{-1}(x)$. They lie in C because $V^2 M \subset p\Lambda^+(M)$ and $pM \subset p\Lambda^+(M)$. Thus $\langle p^{-2} \rangle Fx$ and $\langle p^{-1} \rangle x$ lie in $s(C)$, which is an irreducible component of $\overline{Sh_K^{ss}} \times \overline{\mathbb{F}}$. Observe that $\langle p^{-2} \rangle Fx \in \langle p^{-2} \rangle \mathcal{F}(\tilde{C})$ and $\langle p^{-1} \rangle x \in \langle p^{-1} \rangle \tilde{C}$. We deduce $\langle p^{-2} \rangle \mathcal{F}(\tilde{C}) = s(C) = \langle p^{-1} \rangle \tilde{C}$ because a point of type n lies in a unique irreducible component of $\overline{Sh_K^{ss}} \times \overline{\mathbb{F}}$. \square

Proposition 25. *Let $C \subset p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}}$ be an irreducible component. Then*

$$F \cdot \mathcal{F}(C) = C \cdot F$$

Proof. If $\underline{A}_1 \xrightarrow{f} \underline{A}_0$ is an $\overline{\mathbb{F}}$ -valued point of $p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}}$ whose image lies in C , then $\mathcal{F}(f)$ is the isogeny $\underline{A}_1^{(q)} \xrightarrow{f^{(q)}} \underline{A}_0^{(q)}$ and we have: $f^{(q)} \circ F_{A_1} = F_{A_0} \circ f$, where $F_{A_i} : A_i \rightarrow A_i^{(q)}$ is the Frobenius isogeny. This shows $|F \cdot \mathcal{F}(C)| = |C \cdot F|$. Write X for this support. We define an isomorphism

$$\alpha : p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}} \longrightarrow \left(p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}} \right) \times_{t,s} F$$

by sending $\underline{A}_1 \xrightarrow{f} \underline{A}_0$ to the pair $\left(\underline{A}_1 \xrightarrow{f} \underline{A}_0, \underline{A}_0 \xrightarrow{F} \underline{A}_0^{(q)} \right)$. Similarly, define

$$\beta : p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}} \longrightarrow F \times_{t,s} \left(p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}} \right)$$

by sending $\underline{A}_1 \xrightarrow{f} \underline{A}_0$ to the pair $\left(\underline{A}_1 \xrightarrow{F} \underline{A}_1^{(q)}, \underline{A}_1^{(q)} \xrightarrow{f^{(q)}} \underline{A}_0^{(q)} \right)$. It has degree 1. Consider the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & C \times_{t,s} F \\ \beta \downarrow & & \downarrow c_2 \\ F \times_{t,s} \mathcal{F}(C) & \xrightarrow{c_1} & X \end{array}$$

where c_1 and c_2 are the restriction of c to $F \times_{t,s} \mathcal{F}(C)$ and $C \times_{t,s} F$ respectively. We have $F \cdot \mathcal{F}(C) = \deg(c_1)X$ and $C \cdot F = \deg(c_2)X$. Since α and β have degree 1, we deduce $F \cdot \mathcal{F}(C) = C \cdot F$. \square

We have proved some results on $p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}}$. Observe that diagram (3.1) involves $p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \kappa(\mathcal{O}_{E_p})$. For the relation $H_p(F) = 0$ to make sense, F has to commute with the coefficients of H_p . The pull-back by $p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}} \rightarrow p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \kappa(\mathcal{O}_{E_p})$ defines a \mathbb{Q} -algebra homomorphism

$$(5.1) \quad \mathbb{Q} \left[p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \kappa(\mathcal{O}_{E_p}) \right] \hookrightarrow \mathbb{Q} \left[p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \overline{\mathbb{F}} \right].$$

Corollary 26. *The element F belongs to the centre of $\mathbb{Q} [p\text{-}\mathcal{I} \text{ sog}^{(c)} \times \kappa(\mathcal{O}_{E_p})]$.*

Proof. This follows from proposition 25 using (5.1). \square

5.3. Etale covering. Let K^p and K'^p be two compact open subgroups of $G(\mathbb{A}_f^p)$, such that $K'^p \subset K^p$. Write $K = K_p K^p$ and $K' = K_p K'^p$. Then we have an étale coverings

$$\begin{array}{ccc} \pi : Sh_{K'} & \longrightarrow & Sh_K \\ \Pi : p\text{-}\mathcal{I} \text{ sog}_{K'} & \longrightarrow & p\text{-}\mathcal{I} \text{ sog}_K \end{array}$$

Lemma 27. *The push-forward by Π defines a \mathbb{Q} -algebra homomorphism:*

$$\Pi_* : \mathbb{Q} \left[p\text{-}\mathcal{I} \text{ sog}_{K'}^{(c)} \times k \right] \longrightarrow \mathbb{Q} \left[p\text{-}\mathcal{I} \text{ sog}_K^{(c)} \times k \right]$$

Further, $\Pi_*(F) = \deg(\pi)F$ and $\Pi_*(\langle p \rangle) = \deg(\pi) \langle p \rangle$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} p\text{-}\mathcal{J} \text{ sog}_{K'}^{(c)} \times p\text{-}\mathcal{J} \text{ sog}_{K'}^{(c)} & \xrightarrow{c} & p\text{-}\mathcal{J} \text{ sog}_{K'}^{(c)} \\ \downarrow \Pi \times \Pi & & \downarrow \Pi \\ p\text{-}\mathcal{J} \text{ sog}_K^{(c)} \times_{s,t} p\text{-}\mathcal{J} \text{ sog}_K^{(c)} & \xrightarrow{c} & p\text{-}\mathcal{J} \text{ sog}_K^{(c)} \end{array}$$

If C_1, C_2 are cycles,

$$\Pi_*(C_1 \cdot C_2) = \Pi_* c_*(C_1 \times_{t,s} C_2) = c_*(\Pi \times \Pi)_*(C_1 \times_{t,s} C_2) = c_*(\Pi_*(C_1) \times_{t,s} \Pi_*(C_2)) = \Pi_*(C_1) \cdot \Pi_*(C_2)$$

thus Π_* is a ring homomorphism. We have another commutative diagram:

$$\begin{array}{ccc} p\text{-}\mathcal{J} \text{ sog}_{K'}^{(2)} \times k & \xrightarrow{\Pi} & p\text{-}\mathcal{J} \text{ sog}_K^{(2)} \times k \\ \uparrow F & & \uparrow F \\ \overline{Sh}_{K'} \times k & \xrightarrow{\pi} & \overline{Sh}_K \times k \end{array}$$

thus $\Pi_*(F) = \deg(\pi)F$ and similarly $\Pi_*(\langle p \rangle) = \deg(\pi)\langle p \rangle$. \square

5.4. Main theorem.

Lemma 28. *Let $C \subset p\text{-}\mathcal{J} \text{ sog}^{(c)} \times \overline{\mathbb{F}}$ be a supersingular irreducible component. Assume that the map (s, t) is a closed immersion on $p\text{-}\mathcal{J} \text{ sog}_K^{(c), ss}$. In the ring $\mathbb{Q}[p\text{-}\mathcal{J} \text{ sog} \times \overline{\mathbb{F}}]$, the following relation holds:*

$$C \cdot (F - p^{n-1} \langle p \rangle) = 0$$

Proof. The proof is twofold: First we show that $C \cdot F$ and $C \cdot \langle p \rangle$ have the same support, then we look at multiplicities. The supports $|C \cdot F|$ and $|C \cdot \langle p \rangle|$ are irreducible, of dimension $n - 1$. Indeed, they are the direct images by the composition morphism c of $C \times_{t,s} F$ and $C \times_{t,s} \langle p \rangle$ respectively, which are irreducible. Thus, $|C \cdot F|$ and $|C \cdot \langle p \rangle|$ are irreducible components of $p\text{-}\mathcal{J} \text{ sog}^{(c+2)} \times \overline{\mathbb{F}}$. We clearly have $s(C \cdot F) = s(C \cdot \langle p \rangle)$. Using proposition 24, we have:

$$t(C \cdot F) = \mathcal{F}(C_t) = \langle p \rangle C_t = t(C \cdot \langle p \rangle)$$

Proposition 23 then shows that $|C \cdot F| = |C \cdot \langle p \rangle|$. We denote by X this closed subset.

The projection on C defines isomorphisms $a_F : C \times_{t,s} F \rightarrow C$ and $a_p : C \times_{t,s} \langle p \rangle \rightarrow C$. Write $c_F = c \circ a_F^{-1}$ and $c_p = c \circ a_p^{-1}$. There is a commutative diagram:

$$\begin{array}{ccccc} C \times_{t,s} F & \xrightarrow{c} & X & \xrightarrow{(s,t)} & C_s \times \mathcal{F}(C_t) \\ \uparrow \simeq & \nearrow c_F & & \nearrow \text{id} \times \mathcal{F} & \\ C & \xrightarrow{(s,t)} & C_s \times C_t & & \\ \downarrow \simeq & \searrow c_p & & \searrow \text{id} \times \langle p \rangle & \\ C \times_{t,s} \langle p \rangle & \xrightarrow{c} & X & \xrightarrow{(s,t)} & C_s \times \mathcal{F}(C_t) \end{array}$$

Recall that $\mathcal{F}(C_t) = \langle p \rangle C_t$. By definition, $C \cdot F = \deg(c_F)X$ and $C \cdot \langle p \rangle = \deg(c_p)X$. The diagram shows that $\frac{\deg(c_F)}{\deg(c_p)} = \frac{\deg(\text{id} \times \mathcal{F})}{\deg(\text{id} \times \langle p \rangle)}$. The map $\langle p \rangle : C_t \rightarrow \langle p \rangle C_t$ has degree 1 and $\mathcal{F} : C_t \rightarrow \mathcal{F}(C_t)$ has degree $p^{2\frac{n-1}{2}} = p^{n-1}$ since C_t has dimension $\frac{n-1}{2}$. Thus, $\deg(c_F) = p^{n-1} \deg(c_p)$, and finally $C \cdot F = p^{n-1} C \cdot \langle p \rangle$. \square

Theorem 29. *Let $C \subset p\text{-}\mathcal{J} \text{ sog}^{(c)} \times \overline{\mathbb{F}}$ be an irreducible supersingular component. In the ring $\mathbb{Q}[p\text{-}\mathcal{J} \text{ sog}^{(c)} \times \overline{\mathbb{F}}]$, the following relation holds:*

$$C \cdot (F - p^{n-1} \langle p \rangle) = 0.$$

Proof. Let $K'^p \subset K^p$ such that (s, t) is a closed immersion on $p\text{-}\mathcal{I} \text{ sog}_{K'}^{(c+2)} \times k$ (proposition 19), and C' be a supersingular irreducible component of $p\text{-}\mathcal{I} \text{ sog}^{(c)} \times k$ such that $\Pi(C') = C$. We have $C' \cdot (F - p^{n-1} \langle p \rangle) = 0$ (lemma 28), and taking the image by Π_* , we find $C \cdot (F - p^{n-1} \langle p \rangle) = 0$ (lemma 27). \square

Theorem 30. *Let H_p be the Hecke polynomial. Consider the coefficients of H_p in $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})]$ through $\sigma \circ h$ (see diagram (3.1)). We have the relation:*

$$H_p(F) = 0.$$

Proof. We have $H_p(t) = R(t) \cdot (t - p^{n-1} \langle p \rangle)$ with $R(t) \in \mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})][t]$ (theorem 7). Let $H'_p = \text{cl}(\text{ord}(H_p))$ and $R' = \text{cl}(\text{ord}(R))$. Then $H'_p(t) = R'(t) \cdot (t - p^{n-1} \langle p \rangle)$. Theorem 3 shows that $H'_p(F) = 0$ in $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})]$. Therefore,

$$H_p(F) = (H_p - H'_p)(F) = (R - R')(F) \cdot (F - p^{n-1} \langle p \rangle)$$

The coefficients of H_p and R are linear combinations of supersingular irreducible components of $p\text{-}\mathcal{I} \text{ sog} \times \kappa(\mathcal{O}_{E_p})$. Indeed, they are specialization of cycles of dimension $n - 1$ in $\mathbb{Q}[p\text{-}\mathcal{I} \text{ sog} \times E_p]$, and specialization respects dimensions ([Fulton(1998)], 20.3). These components are either ordinary or supersingular (proposition 18). Thus, the coefficients of $R - R'$ are $n - 1$ -dimensional supersingular cycles, and so is $(R - R')(F)$. Finally, theorem 29 shows that $H_p(F) = 0$. \square

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